ACCELERATED ALGORITHMS OF NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM FOR LAPLACE'S (POISSON'S) EQUATION

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Abstract. In this work some results of modern computer technology mathematically based on Laplace and Poisson's equations are presented. To solve these problems the methods of finite element, Monte Carlo, theory of probability, mathematical physics were used.

Keywords: numerical modelling, elliptic equations, random walks, transitional probability, simplex moving method.

AMS Subject Classification: 65M75, 65N30.

1. Introduction

An effective approach uses a new version of the Monte Carlo method in which a posteriori transition probabilities in the scheme of random walks for the first time are replaced by a priori ones. This leads to a significant acceleration of calculations achieved by the use of a special computational pattern in the form of the simplex element. Successful combination of the Monte Carlo method's probabilistic ideas and bar-centric coordinates of the simplex element allows us not to compile and solve large systems of linear algebraic equations. The traditional application of the finite element's grid in the given area also becomes unnecessary. It is sufficient to provide moving of the simplex which translates boundary information to the test point. Both mathematical models and useful computerized applications which are implementing algorithms of simplex moving method (SMM) were written and obtained. Computerized technology based on SMM and stationary temperature problem and problems of torsion of elastic rods of complex section was developed and implemented [1, 2].

The problem of defining a stationary temperature field within a certain area at a given temperature on the boundary of the region is reduced to the solution of the Dirichlet problem for the Laplace equation.

2. Numerical solution of the Dirichlet problem for Laplace's equation

Let us submit a brief summary of the main results of computer analysis of the problem of stationary temperature distribution in a square plate [3]. This problem with geometrically simple region is attractive because one could compare different computational methods and follow the advantages and disadvantages obtained from them.

Suppose you want to find a stationary temperature distribution in a square plate. In both sides of this plate x = 0 and x = 1 temperature is maintained correspondingly at 0^0 and 100^0. In the side y = 0 temperature increases linearly while in another side y = 1 like a quadratic parabola.

The same task with 25 nodes, 9 of which inner, was solved by application of finite difference method (FDM) followed by solving the resulting system of equations using Cramer's Rule and iterative method. Solution obtained by Cramer's Rule was considered as precise. The same result is achieved on 30 iterations of simultaneous displacement method, on iterations of successive displacement method and on 9 iterations of consistent method of upper relaxation. Almost the same precision for arbitrary point of the given region could be obtained by applying the above mentioned SMM method.

We will apply a standard program for calculating values of the temperature T(A) in the test points $A_1, A_2, \ldots, A_9$. Input data are: coordinates of internal nodes $A_1, A_2, \ldots, A_9$, coordinates of boundary nodes $B_1, B_2, \ldots, B_{12}$, temperature values T(B) at boundary nodes $B_1, B_2, \ldots, B_{12}$. Three simplex templates were used for computations: pattern 1 (B_1, B_4, B_7), pattern 2 (B_1, B_5, B_9) and pattern 3 (B_1, B_6, B_{11}) (Fig 1).

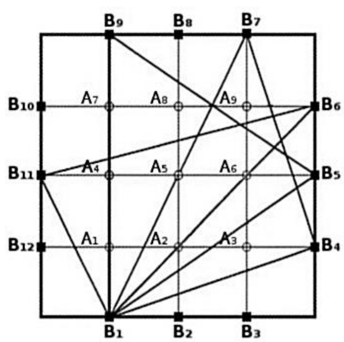

Fig. 1. Points obtained on the plate and computational templates

Computations for each pattern and their average values are given in Table 1. Relative error δ (%) of the numerical data in the research $0.1 \leq \delta \leq 2.5$ which indicates a rather high accuracy of the proposed method.

Tab. 1. Temperature problem solved by using SMM

Test Points	Accurate Solution	SMM Pattern 1	SMM Pattern 2	SMM Pattern 3	SMM Arithmetic Mean	Relative Error δ (%)

A_1	23.493	24.375	22.656	23.958	23.663	0.6
A_2	47.879	47.143	49.143	47.591	47.984	0.2
A_3	73.493	74.375	72.656	73.958	73.663	0.2
A_4	21.094	21.964	19.792	23.177	21.644	2.5
A_5	44.531	44.792	44.792	44.705	44.763	0.5
A_6	71.094	71.964	69.792	73.177	71.644	0.7
A_7	16.350	15.208	17.969	15.625	16.267	0.5
A_8	38.058	37.500	40.365	35.482	37.782	0.7
A_9	66.350	65.208	67.969	65.625	66.267	0.1

The solution for the problem of torsional stress is Prandtl's function which satisfies Poisson's equation and has constant values in the boundary of the given region. Moreover, in a homogeneous region its value in the boundary equals to zero.

3. Numerical solution of the Dirichlet problem for Poisson's equation

Let us consider the problem of torsion of a homogeneous prismatic rod [4]. The region was divided into 20 parts. Figure 2 shows calculated points at which values of a stress function were determined. By means of those values both torque moment and maximum tangent torsion were computed. Dirichlet's problem for Poisson's equation was being solved with the help of grid's method with 31 nodes, 11 of which were internal (Fig 2).

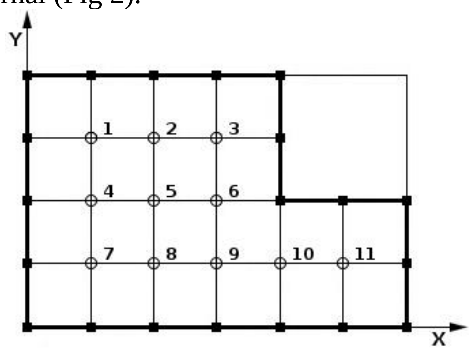

Fig. 2. Shape and cross-section calculated in terms of FDM

The corresponding system of equations was solved by Gauss' method. Since the exact solution for this cross-sectional shape is unknown, the results of calculations were compared with the exact solution for a similar rectangular cross with equal heights. These results were useful for evaluation of the order of stress, while both tensions had the same order. When solving the same problem by finite element method (FEM) the research area was divided into 40 rectangular triangles. On forming matrix of stiffness 29 nodes including 11 interior ones were taken into account. For these nodes the augmented matrix of stiffness was of 11 $\times$ 11 order.

Having compared maximum values of tangent stresses calculated by FDM and FEM we got relative error about 38%, although the values of a function of internal stresses in internal points were equal.

Let us show the way of solving a torsion problem of a noncircular rod cross section (Fig 3) by using the standard program.

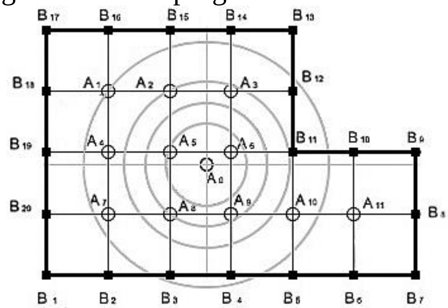

Fig. 3. Shape of the cross section and settlement in terms of SMM

To perform the necessary calculations we must provide the following information:
- the coordinates of research points A(M);
- the coordinates of boundary nodal points B(N);
- the constant value of H required for the simplex element.

These calculations are presented in Table 2.

Tab. 2. Torsion problem solved by using FDM, FEM, SMM

Research Points	**FDM, FEM**	**SMM**
A_0	-	2.65909
A_1	1.390	1.75
A_2	1.780	1.87119
A_3	1.410	1.875
A_4	1.780	1.83333
A_5	2.320	2.46968
A_6	1.860	2.55167
A_7	1.410	2.0202
A_8	1.860	2.06817
A_9	1.710	2.33332
A_{10}	1.123	1.5
A_{11}	0.781	1.25

Having compared the results obtained by SMM and those by alternative methods the following conclusions could be made: SMM defines the value of the stress function for any quantity of randomly located dots or in a single point; numerical results are in a strict correspondence of Prandtl's membrane analogy.

4. Conclusion

Above mentioned methods and results of computer modelling are informative enough so that they can be successfully used in solving problems which are based on mathematical equations of elliptic type that arise in various applications, such as in solving problems of fluid dynamics, heat conduction, theory of elasticity, etc. In particular, two-dimensional Poisson's equation is modelling the following effects:

- Continuous thermal conductivity

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=-q,$$

- Electrostatics

$$\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial Y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon\frac{\partial Y}{\partial y}\right)=-p,$$

- Magnetostatics

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\cdot\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\cdot\frac{\partial A_z}{\partial y}\right)=-j_z.$$

Problems of the calculation of cylindrical tubes under the influence of a uniform internal or external pressure are reduced to problems of planar deformation, which are based on differential equation in partial derivatives of the fourth order of the form,

$$\frac{\partial^4\varphi}{\partial x^4}+2\cdot\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0,$$

where $\varphi(x,y)$ is the function of stress. In polar coordinates this equation can be represented in the following form:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\Theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\Theta^2}\right)=0.$$

It is clear that every solution of the Laplace equation

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\Theta^2}=0$$

is also a solution of the given equation of the fourth order which makes it possible to apply computer technology for solving some problems, e. g. a problem of the stress state of a gas pipeline from corrosion damage.

A new approach to solving boundary value problems of elliptic type offers great opportunities for raising a number of problems in the further theoretical and experimental research and development of modern technologies of computerized implementation.

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